Impact of turbulence on the stratified flow around small particles

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We study the turbulent flow of the density-stratified fluid around a small translating particle. It was found recently [A. M. Ardekani and R. Stocker, Phys. Rev. Lett. **105**, 084502 (2010)] that without turbulence, the familiar Stokes flow is dramatically altered by the stratification. The latter "turns on" the buoyancy introducing a new "cutoff" length scale for the flow around the particle, yielding closed streamlines and a faster decay of velocity. This result, however, didn't account for the potential role of background turbulence, intrinsically present in many aquatic environments, mixing the density and thus opposing the above effect. Here we derive and solve the advection-diffusion equation that describes the interplay of turbulent mixing, diffusion of the stratifying agent and buoyancy. We derive an *exact* expression for fluctuations due to weak background turbulence and show that stronger turbulence can completely change the flow around the particle. Therefore, the account of background turbulence is obligatory in a typical marine environment.

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Recently it was found that density stratification can change the incompressible flow around the small particles strongly [1]. In spite of wide separation of scales of stratification (kilometers) and swimmers ($0.1-1$ mm) in typical marine environment, one cannot neglect the spatial inhomogeneity due to stratification. The reason is that inhomogeneity opens a new pathway of interaction - the buoyancy. The latter has no influence on the incompressible flow when the distribution of the stratifying agent is considered uniform, but it does when inhomogeneity is taken into account. A combination of buoyancy, diffusion (conduction) and viscosity creates a new length-scale L of the order of 1 mm. Beyond that scale the perturbation flow around the particle decays faster than the Stokes flow of the unstratified flow that holds at scales much smaller than L. In sharp contrast with the unstratified case, the flow possesses toroidal eddies and closed streamlines at scales of order L. It was suggested that this may affect propulsion of small organisms and sinking of marine snow particles, diminish the effectiveness of mechanosensing in the ocean [1], stifle nutrient uptake of small motile organisms [2] or potentially hinder the drift-induced biogenic mixing [3].

Clearly, to apply this result one has to determine its domain of validity. However, this effect of stratification should be very sensitive to the presence of a background flow that would redistribute the stratifying agent, altering the buoyancy forces and the resulting flow. In applications this demands, firstly, to include into consideration the background turbulent flow that is present in natural environments invariably. Turbulence mixes the fluid opposing the effect of stratification. Like in the case of stratification, it is incorrect to use the separation of scales of turbulence (the Kolmogorov's scale ℓ_η, see [4],) and of L as an argument to neglect flow inhomogeneity. Turbulence at scales smaller than ℓ_η is a large-scale chaotic flow. In contrast to mixing in the inertial range, turbulence mixes at small scales at a rate independent of the scale [5–7]. That rate is given by the characteristic value λ of the gradient of turbulent velocity field. Note that λ^2 is the energy dissipation per unit mass ϵ divided by the kinematic viscosity ν. The mixing occurs both due to turbulence and diffusion/conduction. Turbulence dominates the mixing at scales larger than the Batchelor scale ℓ_d (where the diffusive time-scale is of order λ^{-1}) and at smaller scales the diffusion/conduction prevails. Thus if turbulence is strong, $\ell_d \lesssim L$, then the flow deviates strongly from the one in the quiescent fluid. In contrast, for weak turbulence $\ell_d \gg L$ the stratified flow [1] holds at the relevant scales smaller or of order L.

This Letter describes consistent, quantitative analysis of the flow around small particles in the presence of both the density stratification and the turbulence. The translating particle distorts the surrounding turbulent flow. This distortion redistributes the stratifying agent, such as temperature, which feeds back the flow through buoyancy. We derive the advection-diffusion equation that describes this interaction by a consistent reduction of the full system of hydrodynamic equations. The main modification relative to the usual equation on the passive scalar field mixed by turbulence [7] is the non-trivial wave-number dependence of the diffusion coefficient. Solving the equation we describe the flow and temperature fields around the moving particle. The solution depends on one dimensionless parameter $\beta \equiv L^2/\ell_d^2$. Turbulence is negligible at $\beta \ll 1$, but at $\beta \gg 1$ the streamlines corresponding to the Stokes' flow without stratification are recovered. It should be emphasized that the results are obtained without modeling the statistics of turbulence and they can be applied to natural environments directly. In the latter, $\beta \gtrsim 1$, where the effect of turbulence is of order one, is typical, so turbulence is important. Thus velocity inhomogeneities at large scales influence the flow around small-scale objects. The relevant parameter is not

the spatial scale ℓ_η, but the temporal scale λ^{-1}.

We use the Boussinesq approximation [9] to describe the interaction of the flow $\boldsymbol{v}$ with the stratified agent θ ,

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\nabla\left(p/\rho\right) + \theta\boldsymbol{g} + \nu\nabla^2\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v}=0,$$
$$\partial_t\theta + \boldsymbol{v}\cdot\nabla\theta = \kappa\nabla^2\theta; \quad \boldsymbol{v}(|\boldsymbol{x}-\boldsymbol{Y}|=a,t)=\boldsymbol{V} \qquad (1)$$

where p/ρ is the pressure divided by the density, $\boldsymbol{g}=-g\hat{z}$ is the gravitational acceleration, $\boldsymbol{Y}$, $\boldsymbol{V}$ and a are the particle's coordinate, velocity and radius respectively [8]). We decompose the flow into the background turbulent flow $\boldsymbol{u}$, P, θ_0 and the perturbation induced by the boundary condition describing the particle, $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) + \boldsymbol{w}'\left[\boldsymbol{x}-\boldsymbol{Y}(t)\right]$, $p/\rho(\boldsymbol{x}) = P_0(\boldsymbol{x}) + P\left[\boldsymbol{x}-\boldsymbol{Y}(t)\right]$ and $\theta(\boldsymbol{x}) = \theta_0(\boldsymbol{x}) + \Theta\left[\boldsymbol{x}-\boldsymbol{Y}(t)\right]$, where $\boldsymbol{w}'$, P, Θ decay at large $\boldsymbol{r}\equiv\boldsymbol{x}-\boldsymbol{Y}(t)$. The perturbations obey, cf. [8],

$$\partial_t\boldsymbol{w}' + \boldsymbol{w}'\cdot\nabla\boldsymbol{w}' + \left[\boldsymbol{u}\left(\boldsymbol{r}+\boldsymbol{Y}\left[t\right]\right) - \boldsymbol{V}(t)\right]\cdot\nabla\boldsymbol{w}' + \sigma\boldsymbol{w}'$$
$$= -\nabla P + \Theta\boldsymbol{g} + \nu\nabla^2\boldsymbol{w}', \quad \partial_t\Theta + \boldsymbol{w}'\cdot\nabla\Theta + \boldsymbol{w}'\cdot\nabla\theta_0$$
$$+\left[\boldsymbol{u}\left(\boldsymbol{r}+\boldsymbol{Y}\left[t\right]\right) - \boldsymbol{V}(t)\right]\cdot\nabla\Theta = \kappa\nabla^2\Theta, \quad \nabla\cdot\boldsymbol{w}'=0, \ (2)$$

where we introduced $\sigma_{ij}(t) = \nabla_j u_i[\boldsymbol{Y}(t),t]$. The boundary conditions are decay far from the particle and $\boldsymbol{w}'(|\boldsymbol{r}|=a) = -(\boldsymbol{u}-\boldsymbol{V})$. We make several assumptions. The experimentally relevant situation corresponds to $L\ll\ell_\eta$, so we assume the latter inequality and study the scales $r\ll\ell_\eta$ (this implies $a\ll\ell_\eta$). Since turbulent velocity is smooth at scales smaller than ℓ_η [4], then we have $\boldsymbol{u}\left(\boldsymbol{r}+\boldsymbol{Y}\left[t\right]\right)\approx\boldsymbol{u}\left(\boldsymbol{Y}\left[t\right]\right)+\sigma\boldsymbol{r}$. Thus $\boldsymbol{u}\left(\boldsymbol{r}+\boldsymbol{Y}\left[t\right]\right) - \boldsymbol{V}(t)\approx\sigma\boldsymbol{r}-\boldsymbol{U}$, where $\boldsymbol{U}\equiv\boldsymbol{V}(t)-\boldsymbol{u}(\boldsymbol{Y}(t),t)$ is the particle's velocity relative to the local flow. We assume $U\ell_\eta/\nu\lesssim 1$ and $U\ell_\eta/\kappa\lesssim 1$ so the terms including $\boldsymbol{U}$ can be dropped from the equations at the scales of interest $r\ll\ell_\eta$. As $\lambda\ell_\eta^2/\nu\sim 1$ holds for turbulence, the first assumption means U is not much larger than the characteristic velocity $u_\eta\sim\lambda\ell_\eta$ of the viscous scale eddies of turbulence [4]. The second assumption gives $U\lesssim u_\eta/Pr$, where $Pr=\nu/\kappa$ is the Prandtl number. As it will become clear below at $Pr\lesssim 1$ turbulence is irrelevant at $L\ll\ell_\eta$. Below we assume $Pr^{1/2}\gg 1$, as it also fits many situations in aquatic environments. Then $U\lesssim u_\eta/Pr$ implies $U\ll u_\eta$. The latter inequality means that during the correlation time λ^{-1} of σ the particle's deviation from the trajectory of the fluid particle is much smaller than ℓ_η. Therefore the statistics of $\sigma(t)$ is the same as the one of the velocity gradient of turbulence in the fluid particle's (Lagrangian) frame.

The above assumptions imply that $Re_{\text{rel}}\sim Ua/\nu$ and Ua/κ are small, which allows to drop all terms quadratic in the perturbation in the equations. Further, we assume that in the domain of interest the correction to the gradients of the stratified agent due to the turbulence is negligible and one can approximate $\nabla\theta_0$ by a constant $-\gamma\hat{z}$ like in the fluid at rest (note that we use the units where θ is dimensionless so $\gamma=(1/\rho_0)(-d\rho/dz)$ has the dimensions of inverse length). Finally we notice that at $r\ll\ell_\eta$ all terms in the LHS of the first of Eqs. (2) can be dropped in comparison with the viscous term. In contrast, the terms $\partial_t\Theta$ and $\sigma\boldsymbol{r}\cdot\nabla\Theta$, that are both of the order $\lambda\Theta$, are comparable with $\kappa\nabla^2\Theta$ at the diffusive scale $\ell_d\equiv(\kappa/\lambda)^{1/2}=\ell_\eta Pr^{-1/2}\ll\ell_\eta$ and are not necessarily negligible. Thus $\partial_t\Theta+(\sigma\boldsymbol{r}\cdot\nabla)\Theta$ should be kept at $r\ll\ell_\eta$. Summarizing the above, we obtain

$$\nabla P = \Theta\boldsymbol{g} + \nu\nabla^2\boldsymbol{w}', \quad \partial_t\Theta + (\sigma\boldsymbol{r}\cdot\nabla)\Theta - \gamma w_z' = \kappa\nabla^2\Theta.$$

Finally, to study the flow at $r\gtrsim L$ where $L\gg a$, one can model the effect of the translating particle on the flow by a point-force term in the momentum equation,

$$\nabla P = \Theta\boldsymbol{g} + \nu\nabla^2\boldsymbol{w}' + f\delta(\boldsymbol{r})\hat{z}, \quad \nabla\cdot\boldsymbol{w}'=0,$$
$$\partial_t\Theta + (\sigma\boldsymbol{r}\cdot\nabla)\Theta - \gamma w_z' = \kappa\nabla^2\Theta, \qquad (3)$$

where $\hat{z}$ is the unit vector in the upward direction [15].

Turbulence is described by $\partial_t\Theta+(\sigma\boldsymbol{r}\cdot\nabla)\Theta$ term which is the usual term describing the advection of the passive scalar fields by turbulence at large Prandtl numbers, see [7] and references therein. This term is comparable with the diffusive term at a characteristic scale $\ell_d=\sqrt{\kappa/\lambda}$ and it is dominating at larger scales. This is the term which effect we study in present work.

We briefly consider the case $L\ll\ell_d$ where to leading order the advection term can be dropped at $r\lesssim L$. This is the case without turbulence considered in [1]. Taking the Laplacian of the first of Eqs. (3) we find,

$$\nabla\nabla^2 P = -(\boldsymbol{g}\gamma/\kappa)w_z' + \nu\nabla^4\boldsymbol{w}' + \nabla^2\boldsymbol{f}\delta(\boldsymbol{r}). \qquad (4)$$

The squared Laplacian term and the buoyancy term $g\gamma w_z'/\kappa$ are of the same order at the scale $L\equiv(\nu\kappa/\gamma g)^{1/4}$ introduced in [1]. At smaller scales viscosity dominates and the Stokes flow holds. At $r\sim L$ a non-trivial change in the flow pattern around the particles occurs [1].

We return to the full system (3). Taking the Fourier transform and using the incompressibility $tr\,\sigma=0$ yields

$$i\boldsymbol{k}P = \Theta\boldsymbol{g} - \nu k^2\boldsymbol{w}' + f\hat{z}, \quad \boldsymbol{k}\cdot\boldsymbol{w}'=0, \qquad (5)$$
$$\partial_t\Theta - (\sigma^t\boldsymbol{k}\cdot\nabla)\Theta - \gamma w_z' = -\kappa k^2\Theta. \qquad (6)$$

We multiply Eq. (5) with $\boldsymbol{k}$ and use the incompressibility condition $i\boldsymbol{k}\cdot\boldsymbol{w}'=0$ to eliminate the pressure,

$$P = igk_z\Theta'/k^2, \quad \Theta'\equiv\Theta-f/g. \qquad (7)$$

Introducing $\hat{k}\equiv\boldsymbol{k}/k$ and the projection $\Pi_{ij}(\boldsymbol{k})$ leads to

$$\nu k^2\boldsymbol{w}' = \Theta'\Pi(\boldsymbol{k})\boldsymbol{g}, \quad \Pi_{ij}(\boldsymbol{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j. \qquad (8)$$

We note that obtaining Eq. (7) involves division by k^2 making it necessary to consider the point $k=0$ separately, see below. Substituting w_z' in Eq. (6), we obtain the following closed advection-diffusion equation for Θ

$$\partial_t\Theta - (\sigma^t\boldsymbol{k}\cdot\nabla)\Theta = -\alpha(\boldsymbol{k})\Theta + \phi(\boldsymbol{k}), \quad \alpha(\boldsymbol{k})\equiv\kappa k^2 d(\boldsymbol{k})$$
$$d(\boldsymbol{k})\equiv 1+\frac{k_\perp^2}{L^4k^6}, \quad \phi(\boldsymbol{k})\equiv\frac{f\gamma k_\perp^2}{\nu k^4}, \qquad (9)$$

where $k_\perp^2 = k^2 - k_z^2$. The stratification produces wave-number dependent diffusion coefficient $d(\boldsymbol{k})$ and the source of the fluctuations ϕ. Although both quantities diverge at $k=0$, the time evolution of $\Theta(\boldsymbol{k})$ at $k\neq 0$ decouples from $k=0$, which is clear from relations below, so one can consider the solution at $k\neq 0$ and then continue it. Note that Eq. (9) has the solution $\Theta(\boldsymbol{k}) = const\times\delta(\boldsymbol{k})$ at zero stratification $\gamma=0$ that describes constant distribution of Θ in the real space. One finds that Θ' obeys

$$\partial_t\Theta' - (\sigma^t\boldsymbol{k}\cdot\nabla)\Theta' = -\alpha(\boldsymbol{k})\Theta' - \kappa k^2 f/g. \quad (10)$$

To find the solution we pass to the moving frame $\tilde{\Theta}(\boldsymbol{k},t) = \Theta'(\boldsymbol{k}(t),t)$ where $\boldsymbol{k}(t)\equiv W^{-1,t}(t)\boldsymbol{k}$ with

$$\dot{W} = \sigma W, \quad \dot{W}^{-1,t} = -\sigma^t W^{-1,t}, \quad W_{ij}(t{=}0){=}\delta_{ij}. \quad (11)$$

Since σ is statistically the same as the velocity gradient of $\boldsymbol{u}$ in the fluid particle's frame, then W is statistically the same as the Jacobi matrix of the turbulent flow backward in time [7]. That is, if we consider the Lagrangian trajectories $\boldsymbol{q}(t,\boldsymbol{r})$ defined by $\partial_t\boldsymbol{q}(t,\boldsymbol{r}) = \boldsymbol{u}[t,\boldsymbol{q}(t,\boldsymbol{r})]$ and $\boldsymbol{q}(t=0,\boldsymbol{r}) = \boldsymbol{r}$, then $W_{ij}(t,\boldsymbol{r}) = \partial_j q_i(t,\boldsymbol{r})$ at $t<0$ describes the evolution of small volumes in the turbulent flow backward in time and obeys Eq. (11). In particular, since the Lyapunov exponents of the backward in time flow are $(-\lambda_3,-\lambda_2,-\lambda_1)$ where $(\lambda_1,\lambda_2,\lambda_3)$ are the Lyapunov exponents of the forward in time flow, then $k(t)$, which is governed by $W^{-1,t}$ rather than $W(t)$, obeys

$$\lim_{t\to-\infty}(1/|t|)\ln[k(t)/k(0)] = \lambda_1, \quad (12)$$

see details in [7]. Thus the growth of $\boldsymbol{k}(t)$ with $|t|$ is similar to the exponential growth of the separation between two infinitesimally close fluid particles in turbulence (governed by the principal Lyapunov exponent λ_1).

The limit in Eq. (12) holds for almost every realization of $\sigma(t)$ and does not involve the randomness of turbulence that disappears after taking the infinite time limit. To describe the fluctuations of $\boldsymbol{k}(t)$ when t is finite, one introduces the polar representation $\boldsymbol{k}(t) = k\exp[\rho(t)]\hat{n}(t)$, where $|\hat{n}|=1$. Using $\dot{\boldsymbol{k}} = -\sigma^t\boldsymbol{k}$ one finds [7]

$$\dot{\hat{n}} = -\sigma^t\hat{n} + \hat{n}\zeta, \quad \dot{\rho} = \zeta, \quad \zeta\equiv -\hat{n}\sigma\hat{n}. \quad (13)$$

It follows that $\ln[k(t)/k] = \int_t^0 \zeta(t')dt'$ where ζ is a finite-correlated noise which correlation time τ_c is of order of the correlation time of σ, so that $\tau_c\sim\lambda^{-1}$. Thus Eq. (12) resembles the law of large numbers. To find the moments of $k(t)$ one introduces

$$\lim_{t\to-\infty}(1/|t|)\ln\langle k^l(t)\rangle \equiv \phi(l). \quad (14)$$

The function $\phi(l)$ is convex. It obeys $\phi(0)=\phi(-3)=0$, so it is negative at $-3<n<0$ and positive otherwise. This holds independently of the statistics of turbulence (see [7] for details). In the moving frame Eq. (10) becomes

$$\partial_t\Theta = -\alpha[\boldsymbol{k}(t)]\tilde{\Theta} - \kappa k^2(t)f/g. \quad (15)$$

We consider Θ at $t=0$, taking the initial condition at $t=-T$ and studying the limit $T\to\infty$, i. e. we focus on the steady state solution. Using $\tilde{\Theta}(t=0) = \Theta'(t=0)$,

$$\Theta' = -\frac{\kappa f}{g}\int_{-\infty}^0 dt\exp\left[-\int_t^0 \alpha\left[\boldsymbol{k}(t')\right]dt'\right]k^2(t). \quad (16)$$

The above together with Eqs. (7)-(8) give implicit solution to the system (3) in the Fourier space. Since for turbulence $\boldsymbol{k}(t)$ is a random vector, then Θ and $\boldsymbol{w}$ are random too and should be studied statistically. The computation of the statistics however cannot be done due to the complex dependence on σ. Thus we consider the limiting cases, which will allow us to understand the behavior of the solution in detail. We introduce new integration variable $s_{\boldsymbol{k}}(t) = \int_t^0 \alpha\left[\boldsymbol{k}(t')\right]dt'$ that depends on $\boldsymbol{k}$ via the final condition $\boldsymbol{k}(0)=\boldsymbol{k}$. We have

$$\Theta'(\boldsymbol{k}) = -\frac{f}{g}\int_0^\infty \frac{\exp[-s]\,ds}{1+k_\perp^2(s)/L^4k^6(s)}, \quad (17)$$

where $\boldsymbol{k}(s)\equiv\boldsymbol{k}[t_{\boldsymbol{k}}(s)]$. We introduce the scaling form $t_{\boldsymbol{k}/L}(s) = \tau_{\boldsymbol{k}}(\beta s)/\lambda$, where $\partial_s\tau_{\boldsymbol{k}} = -k^4(\tau/\lambda)[k_\perp^2(\tau/\lambda) + k^6(\tau/\lambda)]^{-1}$ and $\tau(s=0)=0$, so that

$$s = \int_{\tau_{\boldsymbol{k}}(s)}^0 k^2(t'/\lambda)[1+k_\perp^2(t'/\lambda)/k^6(t'/\lambda)]dt'. \quad (18)$$

Introducing $\boldsymbol{q}(s)\equiv\boldsymbol{k}[\tau_{\boldsymbol{k}}(s)/\lambda]$ one finds

$$\Theta'\left(\frac{\boldsymbol{k}}{L}\right) = -\frac{f}{g}\int_0^\infty F(\beta s)e^{-s}ds, \quad F\equiv\frac{q^6}{q_\perp^2+q^6}. \quad (19)$$

This form of the solution is particularly well-suited for the study of the impact of turbulence because $F(s)$ is determined by turbulence only. If intermittency (dependence of the statistics of σ/λ on the Reynolds number Re) can be neglected, then $F(s)$ varies at scales of order one. The solution depends on one dimensionless parameter, β. Studying how it changes when β is increased from 0 (the stratified flow without turbulence) to ∞ one finds the impact of turbulence. Intermittency is negligible up to rather high Reynolds numbers of 10^5-10^6 due to the smallness of the corresponding anomalous exponents [4], so we consider this case first. Thus $F(\beta s)$ in Eq. (19) varies at scale $1/\beta$. The limit of small β is described by

$$\Theta'(\boldsymbol{k}/L) = -(f/g)\sum_{n=0}^\infty \beta^n F^{(n)}(0), \ \Theta = \Theta' + f/g. \quad (20)$$

To find the solution to order β one can use $-F'(0) = F^2(0)(1/F)'(0) = F^2(0)[q_\perp^2/q^6]'(0)$. Writing the result

in terms of the solution without turbulence $\Theta_0(\boldsymbol{k}/L) = (f_z/g)k_\perp^2\left[k_\perp^2+k^6\right]^{-1}$ [1], one finds

$$\Theta(\boldsymbol{k}/L) = \Theta_0(\boldsymbol{k}/L)\left[1+\delta\Theta(\boldsymbol{k}/L)\right], \tag{21}$$

$$\delta\Theta(\boldsymbol{k}/L) = 2\beta\lambda^{-1}k^8k_\perp^{-2}\left[k^6+k_\perp^2\right]^{-2}\boldsymbol{f}\cdot\sigma^t\boldsymbol{k},$$

where $\boldsymbol{f} \equiv \left[\left(k^2-3k_\perp^2\right)k_x, \left(k^2-3k_\perp^2\right)k_y, -3k_\perp^2 k_z\right]$. Thus in this order the relative correction $\delta\Theta$ to the solution without turbulence is a linear function of the current value of the gradient of the turbulent velocity $\nabla_j u_i$ at the location of the particle. Since the statistics of σ is close to the Lagrangian statistics of $\nabla_j u_i$ (the statistics in the frame of fluid particle), then $\langle\sigma_{ij}\rangle = \langle\nabla_j u_i\rangle = 0$ and $\langle\sigma_{ij}\sigma_{mn}\rangle = \langle\nabla_j u_i\nabla_n u_m\rangle$, where the last average can be taken in Eulerian frame due to incompressibility, giving

$$30\nu\langle\sigma_{ij}\sigma_{mn}\rangle = \epsilon\left[4\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}-\delta_{ij}\delta_{mn}\right]. \tag{22}$$

where isotropy (typically valid for small-scale turbulent fluctuations that determine $\nabla\boldsymbol{u}$ [4]) and spatial uniformity are assumed. The form of $\langle\nabla_j u_i\nabla_n u_m\rangle$ is fixed uniquely by the demands of isotropy, incompressibility and spatial uniformity that implies $\langle\nabla_m u_i\nabla_n u_m\rangle = \langle\nabla_m\left[u_i\nabla_n u_m\right]\rangle = 0$. The relation is *exact* due to stationarity condition $\nu\langle\nabla_j u_i\nabla_j u_i\rangle = \epsilon$. Thus we obtain the *exact* result for fluctuations of Θ (clearly $\langle\delta\Theta\rangle = 0$) around Θ_0 that are caused by the turbulent fluctuations of the background velocity field

$$\langle\delta\Theta^2\rangle^{1/2} = \beta k^8k_\perp^{-2}\left[k^6+k_\perp^2\right]^{-2}\sqrt{[4k^2f^2-2(\boldsymbol{k}\cdot\boldsymbol{f})^2]/15}.$$

The result indicates that the impact of turbulence is of order one at $\beta\sim 1$. When intermittency is taken into account, the higher order terms in the series (20), that holds in the limit $\beta\to 0$ irrespective of the neglect of intermittency, have weight increased by a power of Re. Thus the impact of turbulence is of order one at $\beta\sim \mathrm{Re}^{-\delta}$, where the phenomenological exponent δ is small.

We now consider the limit of strong turbulence $\beta = \lambda(\nu/\kappa\gamma g)^{1/2}\gg 1$ when intermittency is negligible or Re is fixed. One observes that this limit is equivalent to the one of small stratification $\gamma\to 0$ in agreement with consideration that strong mixing opposes stratification canceling its effects completely at $\beta\to\infty$. We consider

$$\frac{g\Theta(\boldsymbol{k}/L)}{f} = \int_0^\infty\frac{q_\perp^2(\beta s)e^{-s}ds}{q_\perp^2(\beta s)+q^6(\beta s)} = \int_0^\infty\frac{q_\perp^2(s)e^{-s/\beta}ds}{\beta[q_\perp^2(s)+q^6(s)]}$$

The RHS is equal to one at $k=0$ recovering $\Theta(0)=f/g$ following directly from Eq. (5). If $k\neq 0$ then the leading order behavior is obtained by setting $\exp[-s/\beta]=1$,

$$\frac{g\Theta(\boldsymbol{k}/L)}{f} = \frac{1}{\beta}\int_0^\infty\frac{q_\perp^2(s)ds}{q_\perp^2(s)+q^6(s)}+o(1/\beta). \tag{23}$$

Returning to the original integration variable, $t = \tau_{\boldsymbol{k}}(s)/\lambda$,

$$\int_0^\infty\frac{q_\perp^2(s)ds}{q_\perp^2(s)+q^6(s)} = \frac{\lambda}{k^2}\int_{-\infty}^0 e^{-2\rho(t)}\left[n_x^2(t)+n_y^2(t)\right]dt.$$

We can use $\langle e^{-2\rho(t)}\rangle\approx\exp[\phi(-2)t]$ following from Eq. (14) at $\lambda t\gg 1$. Since $\hat{n}(t)$ and $\rho(t)$ can be considered as independent, while $\hat{n}$ is distributed isotropically [7], we find $\langle e^{-2\rho(t)}n_i^2(t)\rangle = \exp[\phi(-2)t]/3$. Thus

$$\langle\Theta(\boldsymbol{k}/L)\rangle\sim 2\lambda f/[3\beta gk^2\phi(-2)]. \tag{24}$$

The integral at $k=0$ is convergent due to $\exp[-s/\beta]$ while the integral at $k\sim 1$ is obtained by setting $\exp[-s/\beta]\approx 1$. It can be seen that the transition between the two asymptotic regions is where the two answers are of the same order, that is $\beta k^2\sim 1$. The result in terms of Θ' is particularly simple,

$$\Theta'(\boldsymbol{k})\approx -\frac{f}{g},\ \ k^2\gg\frac{1}{L^2\beta};\ \ |\Theta'(\boldsymbol{k})|\ll\frac{f}{g},\ \ k^2\ll\frac{1}{L^2\beta}.$$

For finding the inverse Fourier transform, one can put $\Theta'\approx -f/g$ at large β uniformly. Substituting $\Theta' = -f/g$ in Eq. (8) recovers the Stokes flow $\nu k^2\boldsymbol{\omega}' = f\Pi(\boldsymbol{k})\hat{z}$.

Thus the solution depends on one dimensionless parameter β so that the Stokes flow with streamlines open everywhere holds at $\beta\gg 1$ and the stratified flow with closed streamlines holds at $\beta\ll 1$. It follows that at $\beta\gtrsim 1$ turbulence cannot be disregarded. In particular, there is a critical value $\beta_c\sim 1$ such that the streamlines open at $\beta=\beta_c$ (there are closed streamlines aty $\beta<\beta_c$ but not at $\beta>\beta_c$. The account of intermittency (the dependence on the Reynolds number) is expected to produce stronger fluctuations of σ and Θ. It should lower the value of β where the account of turbulence is necessary so that in the limit of very high Re turbulence can be important already at $\beta\ll 1$.

Thus turbulence is important at $\beta\gtrsim 1$. It can also be important at $\beta\ll 1$. In this case turbulence is negligible in $\boldsymbol{w}'$, but the total flow $\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{w}'$ can differ from $\boldsymbol{w}'$ significantly due to $\boldsymbol{u}$. The flow $\boldsymbol{u}$ produces the characteristic difference λL of velocities of particles separated by L. The corresponding relative velocity induced by $\boldsymbol{w}'$ is estimated by Ua/L. The ratio of the two differences $\lambda L^2/Ua = (\nu/Ua)(L^2/\ell_\eta^2)$ is the product of the large parameter $\nu/Ua = 1/Re_{\mathrm{rel}}$ and the small parameter L^2/ℓ_η^2. If it is large turbulence is important. Thus turbulence is important at $(\nu/Ua)(L^2/\ell_\eta^2)\gtrsim 1$ or $|\beta|\gtrsim 1$.

Let us now estimate the typical values of β in various aquatic environments. Using the extreme value of the density gradient $\gamma\rho_0 = 1$ kg m^{-4} [1] that may occur locally in fjords [12], lakes and reservoirs [13] with $\mu = 10^{-3}$ kg m^{-1} s^{-1} yields $L\approx 0.6$ mm for salt-stratified water ($\kappa\approx 1.3\times 10^{-9}$ m^2 s^{-1}) and $L\approx 2$ mm for temperature-stratified water ($\kappa\approx 1.4\times 10^{-7}$ m^2 s^{-1}). Further considering weakly turbulent conditions with the dissipation rate per unit mass $\epsilon\approx 10^{-10}$ m^2 s^{-3} (e.g. Kunze et al.[11] measured $\epsilon\lesssim 10^{-9}$ in a coastal inlet) gives $\lambda=\sqrt{\epsilon/\nu}\approx 0.01$ s^{-1}. Thus, the corresponding values of $\beta=\lambda L^2/\kappa$ are ≈ 0.3 and ≈ 2.8 for temperature- and salt-stratified water, respectively. Furthermore, in the

marine environment the buoyancy frequency $N=\sqrt{g\gamma}$ corresponding to the marginal oscillations which the stable stratification supports [14] is typically in the range between 10^{-4} and 10^{-2} s^{-1}, yielding density gradients $\gamma\rho_0$ that ranges between 10^{-6} and 10^{-2} kg m^{-4}, several orders of magnitude lower than that considered in [1]. In some extreme cases, however (e.g. during seasonal thermocline [14]) N may exceed 0.05 s^{-1} so $\gamma\rho_0$ may reach ≈ 0.3 kg m^{-4}. Using this extreme value of density stratification and $\epsilon \approx 10^{-10}$ m^2 s^{-3} we arrive at $\beta \approx 0.5$ and $\beta \approx 5.5$ for temperature- and salt-stratified water, respectively. However for the less extreme conditions of marine turbulence and/or stratification typically $\beta > 1$. For example, for $\epsilon \approx 10^{-9}$ m^2 s^{-3} and $\gamma\rho_0 \approx 0.01$ kg m^{-4} we find $\beta \approx 8$ and $\beta \approx 90$ for temperature and salt stratification, respectively.

We derived and solved the advection-diffusion equation that describes the turbulent flow around small translating particles in the stratified fluid in the limit of large Prandtl numbers. We showed that when intermittency is negligible (which allows very high Re) the solution is determined by one dimensionless parameter β, where turbulence is important when $\beta \gtrsim 1$. Intermittency, important at higher Re, only strengthens the impact of turbulence. We conclude that the account of turbulence is necessary in natural environments.

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[15] We assume here for simplicity, as in [1], that the particle moves only vertically. The transversal motion can be included by introducing transversal component of the force and using the method of superposition. This will not be considered here since it is irrelevant for our conclusions, while making the analysis unnecessarily more cumbersome.